CONFIDENCE INTERVALS OF QUANTILES
FOR THE GENERALIZED LOGISTIC DISTRIBUTION

HONGJOON, SHIN¹  WOOSUNG, NAM²  JUN-HAENG, HEO³

¹ Ph.D. candidate, School of Civil & Environmental Eng., Yonsei University, Seoul, Korea
(Tel : 82-2-393-1597, Fax : 82-2-393-1597, e-mail : sinong@yonsei.ac.kr)
² Ph.D. candidate, School of Civil & Environmental Eng., Yonsei University, Seoul, Korea
(Tel : 82-2-393-1597, Fax : 82-2-393-1597, e-mail : nws77@yonsei.ac.kr)
³ Ph.D. and Professor, School of Civil & Environmental Eng., Yonsei University, Seoul, Korea
(Tel : 82-2-2123-2805, Fax : 82-2-364-5300, e-mail : jhheo@yonsei.ac.kr)

ABSTRACT

Estimation of confidence limits and intervals for the generalized logistic distribution are
presented based on the methods of moments (MOM), maximum likelihood (ML), and
probability weighted moments (PWM). The asymptotic variances of the MOM, ML, and
PWM quantile estimators are derived as a function of the sample size, return period, and
parameters. Such variances can be used for estimating the confidence limits and confidence
intervals of the population quantiles. The formulas obtained do not have simple forms but can
be evaluated numerically.

1. INTRODUCTION

The standard error of estimate accounts for the error due to small samples, but not the error
due to the choice of inappropriate distribution. The standard error of estimate depends in
general on the methods of parameter estimation. Consequently, each method gives a different
standard error of estimate and the most efficient method is that which gives the smallest
standard error of estimate.

There have been several studies related to prediction accuracy. Some of the earlier work
related to confidence intervals has been summarized in Yevjevich (1964). Bobee (1973) used
the distribution to derive the confidence intervals associated with the measures of risk. Kite
(1975) used data generation experiments to derive distributions of extreme events generated
from probability distributions commonly used in hydrology. Hoshi (1981) had suggested an
approximation technique to compute the derivative of a standard gamma quantile with respect
to the shape parameter. This derivative was needed to estimate the sampling variance of a
specified quantile. Heo et al. (2001) derived and compared confidence intervals on population
quantiles for the Weibull model.

The generalized logistic distribution (GLO) was recommended for use with UK flood data
(Institute of Hydrology, 1999). An appealing trait of the GLO distribution is that it is
unbounded above unless the shape parameter is positive. Having an upper limit to a flood
frequency distribution that is close to the maximum observed flow is often unrealistic except
in special situations such as downstream of a large lake. Hence, it is expected the GLO
distribution to be widely used for modeling extremes of natural phenomena and to play an
important role in the frequency analysis.

Consequently, in this paper, we summarize the procedure to estimate the parameters based on
the methods of moments, maximum likelihood, and probability weighted moments and derive
the asymptotic variances of the MOM, ML, and PWM quantile estimators for the GLO
distribution.
2. MODEL DESCRIPTION

(1) GENERALIZED LOGISTIC DISTRIBUTION
The GLO distribution is a generalization of the 2 parameter logistic distribution. It is also a special case of the Kappa distribution. The generalization used here is based on Hosking and Wallis (1997). The cumulative distribution function and the probability density function of the GLO distribution are defined respectively as

\[
F(x) = \left[1 + \left\{1 - k \left(x - \varepsilon \right) / \alpha \right\}^{1/k} \right]^{-1}
\]

\[
f(x) = \left[1 - k \left(x - \varepsilon \right) / \alpha \right]^{1/(k-1)} \left[1 + \left\{1 - k \left(x - \varepsilon \right) / \alpha \right\}^{1/k} \right]^{-2} / \alpha
\]

where, \( \varepsilon \) is the location parameter, \( \alpha \) is the scale parameter, and \( k \) is the shape parameter. The mean, variance, and skewness coefficient of the GLO distribution are given by

\[
\mu = \varepsilon + \alpha \left[1 - \Gamma\left(1 + k\right) / k \right]
\]

\[
\sigma^2 = \alpha^2 \left[\Gamma\left(1 + 2k\right) / k \right] - \Gamma^2\left(1 + k\right) / k^2
\]

\[
C_s = \gamma_s = \mu / \mu_s^{3/2} = k \left(-g_3 + 3g_1^2g_2 - 2g_1^3 \right) / k \left[g_2 - g_1^2 \right]^{3/2}
\]

where, \( g_r = \Gamma\left(1 + rk\right) / \Gamma\left(1 - rk\right) \), \( \Gamma(\cdot) \) is the gamma function.

(2) CONFIDENCE LIMITS ON QUANTILE
In general, the \( \gamma = 1 - \alpha \) confidence limits on a population quantile of return period \( T \) may be determined by

\[
X_{1-\alpha} = \hat{X}_T \pm u_{1-\alpha/2} S_r
\]

where, \( X_{1-\alpha} \) denotes the limits, \( u_{1-\alpha/2} \) is the \( 1 - \alpha / 2 \) quantile of the standard normal distribution, \( \hat{X}_T \) is the quantile estimator, and \( S_r \) is the standard deviation of \( \hat{X}_T \) or the standard error. The standard error in Eq. (6) depends on the type of distribution and on the estimation procedure used to obtain the parameters of the distribution. And the quantile estimator can be written as \( \hat{X}_T = g(\hat{\varepsilon}, \hat{\alpha}, \hat{k}) \), then the asymptotic variance of \( \hat{X}_T \) may be expressed as

\[
S^2_r = \left(\frac{\partial X_T}{\partial \varepsilon}\right)^2 \text{Var}(\hat{\varepsilon}) + \left(\frac{\partial X_T}{\partial \alpha}\right)^2 \text{Var}(\hat{\alpha}) + \left(\frac{\partial X_T}{\partial k}\right)^2 \text{Var}(\hat{k})
\]

\[
+ 2 \left(\frac{\partial X_T}{\partial \varepsilon}\right) \left(\frac{\partial X_T}{\partial \alpha}\right) \text{Cov}(\hat{\varepsilon}, \hat{\alpha}) + 2 \left(\frac{\partial X_T}{\partial \varepsilon}\right) \left(\frac{\partial X_T}{\partial k}\right) \text{Cov}(\hat{\varepsilon}, \hat{k}) + 2 \left(\frac{\partial X_T}{\partial \alpha}\right) \left(\frac{\partial X_T}{\partial k}\right) \text{Cov}(\hat{\alpha}, \hat{k})
\]

3. ESTIMATION OF QUANTILES
The cumulative distribution function of \( x \) can be written in the inverse form \( x = x(F) \) and by substituting \( F = 1 - 1/T \), the \( T \)-year quantile estimator is given by Eq. (8).

\[
\hat{x}_T = \varepsilon + \alpha \left\{1 - (T-1) \right\}^{1/k}
\]

Also, the estimator \( \hat{x}_T \) may be written in terms of the sample mean \( \hat{\mu} \), sample standard deviation \( \hat{\sigma} \), and the frequency factor \( \hat{K}_T \) as (Chow, 1951)

\[
\hat{x}_T = \hat{\mu} + \hat{K}_T \hat{\sigma}
\]
(1) METHOD OF MOMENTS

The skewness coefficient of the GLO model is given by

\[ C_s = \gamma_1 = \frac{\mu_3/\mu_2^{3/2}}{\left[\Gamma(1+2k)\Gamma(1-2k)-\Gamma^2(1+k)^2(1-k)^2\right]^{1/2}} \]  

(10)

An approximate solution of Eq. (11) is given as followings(Rao and Hamed, 2000).

For \(-10 < C_s < 10\) and \(-1/3 < k < 1/3\), \(k = 2\tan^{-1}(-0.59484C_s)/3\pi\).

For \(0 < C_s < 10\) and \(1/3 < k < 1/2\), \(k = \tan^{-1}(0.03688-0.29824C_s)/3\pi+0.5\).

For \(-10 < C_s < 0\) and \(-1/2 < k < -1/3\), \(k = \tan^{-1}(-0.036884-0.29824C_s)/3\pi-0.5\).

However, more precise shape parameter estimate can be found by using the numerical method such as the Newton-Raphson procedure. For this purpose, Eq. (11) is rewritten as

\[ G(\hat{k}) = \frac{k}{(\hat{g}_3 + 3\hat{g}_2\hat{g}_1 - 2\hat{g}_1^3)} \left(\frac{3\hat{g}_2\hat{d}_2 + 3\hat{d}_2 \hat{g}_1 - \hat{d}_1 - 6\hat{g}_2^2\hat{d}_1}{(\hat{g}_2 - \hat{g}_1^2)^{3/2}}\right) - \frac{3\hat{k}}{2\left(\hat{g}_2 - \hat{g}_1^2\right)^{3/2}} \left(\frac{(3\hat{g}_2\hat{g}_1 - \hat{g}_1^3)(\hat{d}_2 - 2\hat{g}_1\hat{d}_1)}{(\hat{g}_2 - \hat{g}_1^2)^{3/2}}\right) \]  

(13)

where, \(\hat{g}_r = \Gamma(1+r\hat{\beta})\Gamma(1-r\hat{\beta})\) and \(\hat{d}_r = dg_r / dr\hat{k} = r g_r[\psi(1+r\hat{k}) - \psi(1-r\hat{k})]\), \(\Gamma()\) and \(\psi()\) are the gamma and digamma functions, respectively. The solution for \(\hat{k}\) can be obtained by using numerical method such as Newton’s method. The iteration is repeated until \(G(\hat{k})\) is sufficiently close to zero. Once \(\hat{k}\) is determined, \(\alpha\) and \(\varepsilon\) are obtained from Eqs. (3) and (4) as

\[ \hat{\alpha} = \frac{\alpha^2\hat{k}^2}{\left[\Gamma(1+2\hat{k})\Gamma(1-2\hat{k})-\Gamma^2(1+k)^2(1-k)^2\right]^{1/2}} \]  

(14)

\[ \hat{\varepsilon} = \hat{\mu} - \frac{\alpha}{\hat{k}} \left[1 - \Gamma(1+\hat{k})\Gamma(1-\hat{k})\right] \]  

(15)

(2) METHOD OF MAXIMUM LIKELIHOOD

The log-likelihood function of the generalized logistic distribution is given by

\[ LL = -N\ln(\alpha) + \left(\frac{1}{\beta} - 1\right)\sum_{i=1}^{N} \ln \left[1 - \frac{k}{\alpha}(x_i, -\varepsilon)\right] - 2\sum_{i=1}^{N} \ln \left[\left(1 - \frac{k}{\alpha}(x_i, -\varepsilon)\right)^{1/4}\right] \]  

(16)

Taking partial derivatives of the log-likelihood function of Eq. (16) with respect to \(\varepsilon\), \(\alpha\), and \(k\), respectively, and equating them to zero (Rao and Hamed, 2000)

\[ -\frac{\partial LL}{\partial \varepsilon} = Q / \alpha = 0 \]  

(17)

\[ -\frac{\partial LL}{\partial \alpha} = \left(\frac{P + Q}{\alpha k}\right) / \alpha k = 0 \]  

(18)

\[ -\frac{\partial LL}{\partial k} = \left[R - \left(\frac{P + Q}{k}\right)\right] / k = 0 \]  

(19)

\[ P = N - 2\sum_{i=1}^{N} e^{-\gamma_i} \left(1 + e^{-\gamma_i}\right)^{-1} \]  

(20)

\[ Q = (k-1) \sum_{i=1}^{N} e^{\gamma_i} + 2\sum_{i=1}^{N} e^{\gamma_i} \left(1 + e^{-\gamma_i}\right)^{-1} \]  

(21)
$R = N - \sum_{i=1}^{N} y_i + 2 \sum_{i=1}^{N} y_i e^{-y_i} \left(1 + e^{-y_i}\right)^{-1}$ \hfill (22)

where, $y_i = -k^{-1} \ln \left[1 - k \left(x_i - \varepsilon\right)/\alpha\right]$

The second partial derivatives of GLO distribution are given in Appendix A. The ML estimators are obtained by solving Eqs. (20) to (22) using the iteration in Eq. (23), and the iteration is repeated until Eqs. (17) to (19) are sufficiently close to zero.

\[
\begin{bmatrix}
\varepsilon_{n+1} \\
\alpha_{n+1} \\
\kappa_{n+1}
\end{bmatrix} = \begin{bmatrix}
\varepsilon \\
\alpha \\
k
\end{bmatrix} + \begin{bmatrix}
d\varepsilon_n \\
d\alpha_n \\
dk_n
\end{bmatrix}
\] \hfill (23)

where,

\[
\begin{bmatrix}
d\varepsilon_n \\
d\alpha_n \\
dk_n
\end{bmatrix} = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \varepsilon^2} & -\frac{\partial^2 \log L}{\partial \varepsilon \partial \alpha} & -\frac{\partial^2 \log L}{\partial \varepsilon \partial \kappa} \\
-\frac{\partial^2 \log L}{\partial \alpha \partial \varepsilon} & -\frac{\partial^2 \log L}{\partial \alpha^2} & -\frac{\partial^2 \log L}{\partial \alpha \partial \kappa} \\
-\frac{\partial^2 \log L}{\partial \kappa \partial \varepsilon} & -\frac{\partial^2 \log L}{\partial \kappa \partial \alpha} & -\frac{\partial^2 \log L}{\partial \kappa^2}
\end{bmatrix}^{-1} \begin{bmatrix}
\partial \log L \\
\partial \varepsilon \\
\partial \alpha \\
\partial \kappa
\end{bmatrix}
\] \hfill (24)

### (3) METHODS OF PROBABILITY WEIGHTED MOMENTS

The population probability weighted moments of the GLO distribution is given by Eq.(25).

\[
M_{r,s} = \int_0^\infty \left(\varepsilon + \frac{\alpha}{k} \frac{\alpha}{y^k}\right)(1+y)^{r-s} \frac{1}{\alpha} y^{1-k} (1+y)^{2-k} \left(-\alpha\right) y^{k-1} dy
\]

\[
= \frac{1}{(r+1)} \left(\varepsilon + \frac{\alpha}{k}\right) - \frac{\alpha}{k} B(k+1,1+r-k) = \frac{1}{(r+1)} \left(\varepsilon + \frac{\alpha}{k}\right) - \frac{\alpha}{k} \frac{\Gamma(k+1)\Gamma(1+r-k)}{\Gamma(2+r)}
\]

where, $y = \left[1 - k \left(x - \varepsilon\right)/\alpha\right]^{1/k}$

The general form of the population PWM of the GLO distribution is given by Eq. (26).

\[
B_r = \left(\varepsilon + \alpha / k\right) / (r+1) - \alpha \Gamma(k+1)/\kappa \Gamma(2+r)
\] \hfill (26)

By substituting these three population PWMs by the corresponding sample PWMs, $\hat{B}_0$, $\hat{B}_1$, and $\hat{B}_2$, the PWM estimator of the shape parameter $k$ is a solution of Eq. (27).

\[
\frac{3\hat{B}_2 - \hat{B}_0}{2\hat{B}_1 - \hat{B}_0} = 0.5\Gamma(1+k)\Gamma(3-k) + \Gamma(1+k)\Gamma(1-k) = \frac{\Gamma(3-k) - 2\Gamma(1-k)}{2\Gamma(2-k) - 2\Gamma(1-k)}
\] \hfill (27)

Equation can be solved numerically for $k$ and the PWM estimators of the parameters $\alpha$ and $\varepsilon$ may be obtained as

\[
\hat{\alpha} = \hat{k} \left(2\hat{B}_1 - \hat{B}_0\right) / \Gamma(1+k) \{\Gamma(1-k) - \Gamma(2-k)\}
\] \hfill (28)

\[
\hat{\varepsilon} = \hat{B}_0 - \hat{\alpha} / \hat{k} \left[1 - \Gamma(1+k)\Gamma(1-k)\right]
\] \hfill (29)

### 4. CONFIDENCE INTERVALS ON QUANTILES

The $1 - \delta$ confidence interval $X_{1-\delta}$ on the population quantiles may be approximated by

\[
\hat{X}_{1-\delta} = \bar{X} \pm u_{1-\delta/2} \hat{S} \hat{r}
\] \hfill (30)
(1) STANDARD ERROR BY MOMENTS

By using the first three sample moments, the variance of $T_X$ can be written as Kite (1988)

$$S_2^2 = \left( \frac{\mu}{N} \right)^2 \left[ 1 + K_i \gamma_i + \frac{K_i}{4} \left( \gamma_i - 1 \right) + \frac{(\partial K_i)}{(\partial \gamma_i)} \left\{ 2 \gamma_i - 3 \gamma_i^2 - 6 + K_i \left( \gamma_i - 6 \gamma_i^2 - 10 \gamma_i \right) \right\} \right]$$

$$+ \left( \frac{(\partial K_i)}{(\partial \gamma_i)} \right)^2 \left\{ \gamma_i - 3 \gamma_i^2 - 6 \gamma_i + \frac{9}{4} \gamma_i^2 \gamma_i + \frac{35}{4} \gamma_i^2 \right\}$$

where, $\gamma_i = \mu_i / \mu_i^{1/2} = k / k \left( -g_3 + 3g_3g_2 - g_3^2 \right)^{3/2}$

$$\gamma_2 = \mu_2 / \mu_2^{1/2} = k / k \left( -g_3 + 3g_3^2 - 4g_3g_2 + 6g_3^2g_2 \right) \left( g_2 - g_2^2 \right)$$

$$\gamma_3 = \mu_3 / \mu_3^{1/2} = k / k \left( -g_3 + 10g_3^3g_2 - 10g_3^3g_2 - 4g_3^2g_2 - 5g_3g_2 \right) \left( g_2 - g_2^2 \right)^{3/2}$$

$$\gamma_4 = \mu_4 / \mu_4^{1/2} = k / k \left( g_2 + 15g_2^3g_4 - 20g_4^3g_4 + 15g_2^3g_4 - 5g_2^2g_4 - 6g_2g_4 \right) \left( g_2 - g_2^2 \right)^{-3/2}$$

In addition, the derivative of $K_T$ with respect to $\gamma_i$ can be written as

$$\frac{\partial K_T}{\partial \gamma_i} = \left( \frac{\partial K_T}{\partial k} \right) \left( \frac{\partial k}{\partial \gamma_i} \right)$$

The two partial derivatives in the right hand side of Eq. (32) can be determined as follows. From Eq. (10), the derivative of $K_T$ with respect to $k$ and the derivative of $\gamma_i$ with respect to $k$ are

$$\frac{\partial K_T}{\partial k} = k \left[ \frac{d_i + (T-1)^k \log(T-1) - \frac{1}{2} \left( g_2 - g_2^2 \right)^2}{(g_2 - g_2^2)^2} \right]$$

$$\frac{\partial \gamma_i}{\partial k} = k \left[ \frac{3d_i (g_2 + 3g_2^2 - 6g_2^2)}{(g_2 - g_2^2)^2} - \frac{3}{2} \left( g_2 - g_2^2 \right)^2 \right]$$

where, $d_i = r \cdot g_i \psi(1 + rk) - \psi(l - rk)$, and the standard error can therefore be evaluated by numerically evaluating its components and substituting them into Eq. (31).

(2) STANDARD ERROR BY MAXIMUM LIKELIHOOD

The variance of the ML estimator of quantiles for the GLO distribution can be obtained from Eq. (8). The derivatives of $X_T$ with respect to the parameters $\varepsilon, \alpha$, and $k$ are Eqs. (36) to (38). On the other hand, the variance and covariance terms for ML estimators are the elements of the inverse of information matrix in Eq. (35). Hence

$$\frac{\partial \hat{\varepsilon}}{\partial \hat{\varepsilon}} = 1$$

$$\frac{\partial \hat{\varepsilon}}{\partial \hat{\alpha}} = 1 - (T-1)^{-1}$$

$$\frac{\partial \hat{\varepsilon}}{\partial \hat{k}} = 1 - (T-1)^{-1}$$
\[ \frac{\partial \hat{\alpha}}{\partial \hat{k}} = -\alpha / \hat{k}^2 [1 - (T - 1)^{-\hat{k}}] + \alpha / \hat{k} (T - 1)^{-\hat{k}} \log(T - 1) \]  

(38)

Thus, substituting Eqs. (36) to (38), \( \text{Var}(\hat{\epsilon}), \text{Var}(\hat{\alpha}), \text{Var}(\hat{k}), \text{Cov}(\hat{\epsilon}, \hat{\alpha}), \text{Cov}(\hat{\alpha}, \hat{k}), \) and \( \text{Cov}(\hat{\epsilon}, \hat{k}) \) into Eq. (7) yields \( S^2_T \).

(3) STANDARD ERROR BY PROBABILITY WEIGHTED MOMENTS

The variance of the PWM quantile estimator \( \hat{X}_r \) for the GLO distribution is obtained by replacing the parameter estimators. From Eq. (8) the derivatives of \( \hat{X}_r \) with respect to the parameters \( \epsilon, \alpha, \) and \( k \) are respectively Eqs. (36) through (38).

The terms \( \text{Var}(\hat{\epsilon}), \text{Var}(\hat{\alpha}), \text{Var}(\hat{k}), \text{Cov}(\hat{\epsilon}, \hat{\alpha}), \text{Cov}(\hat{\alpha}, \hat{k}), \text{Cov}(\hat{\epsilon}, \hat{k}) \) are given by

\[
\text{Var}(\hat{\epsilon}) = F_0^2 B_{00} + F_1^2 B_{11} + F_k^2 E^2 C + 2 F_0 F_1 B_{01} + 2 F_0 F_k E C_0 + 2 F_1 F_k E C_1 \tag{39}
\]

\[
\text{Var}(\hat{\alpha}) = G_0^2 B_{00} + G_1^2 B_{11} + G_k^2 E^2 C + 2 G_0 G_1 B_{01} + 2 G_0 G_k E C_1 + 2 G_1 G_k E C_0 \tag{40}
\]

\[
\text{Var}(\hat{k}) = E^2 C \tag{41}
\]

\[
\text{Cov}(\hat{\epsilon}, \hat{\alpha}) = F_0 (G_0 B_{00} + G_1 B_{01} + G_k E C_0) + F_1 (G_0 B_{01} + G_1 B_{11} + G_k E C_1) \tag{42}
\]

\[
\text{Cov}(\hat{\epsilon}, \hat{k}) = F_0 E C_0 + F_1 E C_1 + F_k E^2 C \tag{43}
\]

\[
\text{Cov}(\hat{\alpha}, \hat{k}) = G_0 E C_0 + G_1 E C_1 + G_k E^2 C \tag{44}
\]

And the terms in Eqs. (39) through (44) are also given in Appendix B.

Finally, the asymptotic variance of \( \hat{X}_r \) for the GLO distribution becomes

\[
S^2_T = \frac{1}{T^2} \left[ \text{Var}(\epsilon) + \text{Var}(\alpha) \left( \frac{1 - (T - 1)^{-\hat{k}}}{\hat{k}} \right)^2 + \text{Var}(k) \left( -\frac{\hat{\alpha}}{\hat{k}^2} \left( \frac{1 - (T - 1)^{-\hat{k}}}{\hat{k}} \right)^2 + \frac{\hat{\alpha}}{\hat{k}} (T - 1)^{-\hat{k}} \log(T - 1) \right)^2 \right] \tag{45}
\]

5. CONCLUSIONS

Estimation techniques for determining the confidence intervals for the GLO distribution are presented based on the method of moments (MOM), method of maximum likelihood (ML), and method of probability weighted moments (PWM). All three estimation methods require an iterative or numerical solution to estimate the shape parameter. The asymptotic variances of the MOM, ML, and PWM quantile estimators for the GLO distribution are derived as a function of the sample size, return period, and parameters, such variances can be used for estimating the confidence limits and confidence intervals of the population quantiles. The formulas obtained do not have simple forms but can be evaluated numerically.

REFERENCES


Appendix A. The second partial derivatives of the log-likelihood function for the GLO distribution

The first order partial derivatives are given by Eqs. (17) to (19) and the second partial derivatives are given as

\[
\frac{\partial^2 \log L}{\partial \varepsilon^2} = \frac{1}{\alpha} \frac{\partial Q}{\partial \varepsilon} - \frac{1}{\alpha^2} \frac{\partial Q}{\partial \alpha} - \frac{1}{\alpha} \frac{\partial Q}{\partial \alpha} \frac{\partial Q}{\partial \varepsilon} - \frac{1}{\alpha \varepsilon} \frac{\partial Q}{\partial \alpha} \frac{\partial Q}{\partial \varepsilon}
\]
\[\text{(A1)}\]

\[
\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{1}{\alpha} \frac{\partial P}{\partial \alpha} - \frac{1}{\alpha} \frac{\partial P}{\partial \alpha} \frac{\partial P}{\partial \alpha} - \frac{1}{\alpha} \frac{\partial P}{\partial \alpha} \frac{\partial P}{\partial \alpha} \frac{\partial P}{\partial \alpha} - \frac{1}{\alpha \alpha} \frac{\partial P}{\partial \alpha} \frac{\partial P}{\partial \alpha} \frac{\partial P}{\partial \alpha}
\]
\[\text{(A9)}\]

\[
\frac{\partial^2 \log L}{\partial \varepsilon \partial \alpha} = \frac{1}{\alpha} \frac{\partial R}{\partial \varepsilon} - \frac{1}{\alpha} \frac{\partial R}{\partial \varepsilon} \frac{\partial R}{\partial \alpha} - \frac{1}{\alpha} \frac{\partial R}{\partial \varepsilon} \frac{\partial R}{\partial \alpha} \frac{\partial R}{\partial \alpha} - \frac{1}{\alpha \alpha} \frac{\partial R}{\partial \varepsilon} \frac{\partial R}{\partial \alpha} \frac{\partial R}{\partial \alpha} \frac{\partial R}{\partial \alpha}
\]
\[\text{(A10)}\]

\[
\frac{\partial^2 \log L}{\partial \varepsilon \partial k} = -\frac{1}{k^2} \left( R - \frac{P + Q}{k} \right) + \frac{1}{k} \left[ \frac{\partial R}{\partial \varepsilon} - \frac{1}{\alpha} \frac{\partial P}{\partial \alpha} + \frac{1}{k} \frac{\partial Q}{\partial \alpha} \frac{\partial Q}{\partial \varepsilon} + \frac{P + Q}{k^2} \right]
\]
\[\text{(A11)}\]

where,

\[
\frac{\partial P}{\partial \varepsilon} = 2 \left[ \sum_{i=1}^{n} \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \varepsilon} - \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \alpha} \frac{\partial y_i}{\partial \varepsilon} - \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \alpha} \frac{\partial y_i}{\partial \varepsilon} \right]
\]
\[\text{(A12)}\]

\[
\frac{\partial Q}{\partial \alpha} = k(k-1) \sum_{i=1}^{n} e^{y_i} \frac{\partial y_i}{\partial \alpha} + 2 \sum_{i=1}^{n} (k-1) e^{y_i} \frac{\partial y_i}{\partial \alpha} \frac{\partial y_i}{\partial \alpha} + \sum_{i=1}^{n} e^{y_i} \frac{\partial y_i}{\partial \alpha} \frac{\partial y_i}{\partial \alpha} \frac{\partial y_i}{\partial \alpha}
\]
\[\text{(A13)}\]

\[
\frac{\partial R}{\partial \varepsilon} = \sum_{i=1}^{n} \left( \frac{\partial y_i}{\partial \varepsilon} \right) + 2 \left[ \sum_{i=1}^{n} \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \varepsilon} - \sum_{i=1}^{n} y_i \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \varepsilon} + \sum_{i=1}^{n} y_i \frac{1}{e^{\gamma_i} (1 + e^{\gamma_i})} \frac{\partial y_i}{\partial \varepsilon} \right]
\]
\[\text{(A15)}\]
\[ \frac{\partial R}{\partial \alpha} = \sum_{i=1}^{n} \left( \frac{\partial y_i}{\partial \alpha} \right) + 2 \left[ \sum_{i=1}^{n} \frac{1}{e^{y_i} - 1} \frac{\partial y_i}{\partial \alpha} - \sum_{i=1}^{n} y_i e^{y_i} \frac{\partial y_i}{\partial \alpha} + \sum_{i=1}^{n} y_i e^{y_i} \frac{\partial y_i}{\partial \alpha} \right] \]  

(A16)

\[ \frac{\partial R}{\partial k} = \sum_{i=1}^{n} \left( \frac{\partial y_i}{\partial k} \right) + 2 \left[ \sum_{i=1}^{n} \frac{1}{e^{y_i} - 1} \frac{\partial y_i}{\partial k} - \sum_{i=1}^{n} y_i e^{y_i} \frac{\partial y_i}{\partial k} + \sum_{i=1}^{n} y_i e^{y_i} \frac{\partial y_i}{\partial k} \right] \]  

(A17)

\[ \frac{\partial y_i}{\partial \varepsilon} = -\frac{1}{\alpha} e^{\varepsilon} \quad \frac{\partial y_i}{\partial \alpha} = -\frac{1}{\alpha k} \left( e^{\varepsilon} - 1 \right) \quad \frac{\partial y_i}{\partial k} = -\frac{y_i}{k} + \frac{1}{k^2} \left( e^{\varepsilon} - 1 \right) \]  

(A18)-(A20)

**Appendix B. Derivation of the variance of the quantile estimator, \( S_T^2 \) based on PWM method**

The quantile estimator of \( \hat{T}_X \) can be written as a function of the first three sample PWMs and \( \hat{T}_X \) can be obtained by

\[
S_T^2 = \left( \frac{\partial x}{\partial B_0} \right)^2 Var(\hat{B}_0) + \left( \frac{\partial x}{\partial B_1} \right)^2 Var(\hat{B}_1) + \left( \frac{\partial x}{\partial B_2} \right)^2 Var(\hat{B}_2) \\
+ 2 \left( \frac{\partial x}{\partial B_0} \right) \left( \frac{\partial x}{\partial B_1} \right) Cov(\hat{B}_0, \hat{B}_1) + 2 \left( \frac{\partial x}{\partial B_0} \right) \left( \frac{\partial x}{\partial B_2} \right) Cov(\hat{B}_0, \hat{B}_2) + 2 \left( \frac{\partial x}{\partial B_1} \right) \left( \frac{\partial x}{\partial B_2} \right) Cov(\hat{B}_1, \hat{B}_2)
\]  

(B1)

However, the quantile estimator cannot be explicitly written as a function of the sample PWMs because the shape parameter estimator is implicitly expressed as a function of the sample PWMs. Therefore, the variance-covariance matrix of the sample PWMs should be obtained firstly. Then this matrix is transformed to the variance-covariance matrix of the parameter estimators by using a Jacobian transformation. Finally, the variance of the quantile estimator can be obtained. The elements \( B_{rs} \) of variance-covariance matrix \( B \) are given by

\[
B_{rs} = J_{rs} + J_{wr}
\]  

(B2)

\[
J_{rs} = \iint_{x,y} \{ F(x) \}^r \{ F(y) \}^s F(x) \{ 1 - F(y) \} dx dy
\]  

(B3)

For the GLO distribution,

\[
J_{rs} = \alpha^2 \int_0^1 u^{-r} v^{-s-1} (1-u)^{k-1} (1-v)^k du dv
\]

\[
= \alpha^2 \int_0^1 u^{-r} (1-u)^{k-1} (1-v)^{-s-1} (1+k)^{s} F_1(1,1+s;2+k;1-u) du
\]

\[
= \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(r+s+1-2k) \Gamma(r+s+2) \Gamma F_1(1,1+s,1+2k;1,1+s,1+2k;1,1+s,1+2k;2,2+k;2,2+k;1)
\]

where, \( \Gamma(\cdot) \) is the gamma function, \( _3F_2 \) is the generalized hypergeometric function of unit argument, \( u = F(x) = \left[ 1 + \left\{ 1-k \left( x-\varepsilon / \alpha \right) \right\}^{1/k} \right]^{-1} \), \( v = F(y) = \left[ 1 + \left\{ 1-k \left( x-\varepsilon / \alpha \right) \right\}^{1/k} \right]^{-1} \).

\[
J_{00} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(1-2k) \Gamma(2)^{-s}_3F_2(1,1,1+2k;1,1+2k;2,2+k;1)
\]

(B5)

\[
J_{11} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(3-2k) \Gamma(4)^{-s}_3F_2(1,2,1+2k;4,2+2+k;1)
\]

(B6)

\[
J_{22} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(5-2k) \Gamma(6)^{-s}_3F_2(1,3,1+2k;6,2+2+k;1)
\]

(B7)

\[
J_{01} = J_{10} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(2-2k) \Gamma(3)^{-s}_3F_2(1,2,1+2k;3,2+k;1)
\]

(B8)

\[
J_{02} = J_{20} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(3-2k) \Gamma(4)^{-s}_3F_2(1,3,1+2k;4,2+k;1)
\]

(B9)

\[
J_{12} = J_{21} = \alpha^2 (1+k)^{-s} \Gamma(1+2k) \Gamma(4-2k) \Gamma(5)^{-s}_3F_2(1,3,1+2k;5,2+k;1)
\]

(B10)

Therefore, the elements of the variance-covariance matrix \( B \) are given by
For the GLO distribution, the asymptotic variance of the PWM estimator of quantile, \( \hat{x}_r \) can be found by using successively the following transformations.

\[
\begin{bmatrix}
\hat{B}_0 \\
\hat{B}_1 \\
\hat{B}_2 \\
R
\end{bmatrix} \rightarrow \begin{bmatrix}
\hat{B}_0 \\
\hat{B}_1 \\
\hat{B}_2 \\
\alpha \\
k
\end{bmatrix} \rightarrow \left[ \hat{x}_r \right]
\] \hspace{1cm} \text{(B11)}

The shape parameter estimator is a function of the sample PWMs, but cannot be written explicitly, thus we need a transformation. Additional details of the transformations follow.

1) 1ST TRANSFORMATION

For the 1st transformation, the Jacobian matrix is given by

\[
J_1 = \begin{bmatrix}
\partial \hat{B}_0 / \partial B_0 & \partial \hat{B}_0 / \partial B_1 & \partial \hat{B}_0 / \partial B_2 & \partial \hat{B}_0 / \partial R \\
\partial \hat{B}_1 / \partial B_0 & \partial \hat{B}_1 / \partial B_1 & \partial \hat{B}_1 / \partial B_2 & \partial \hat{B}_1 / \partial R \\
\partial \hat{B}_2 / \partial B_0 & \partial \hat{B}_2 / \partial B_1 & \partial \hat{B}_2 / \partial B_2 & \partial \hat{B}_2 / \partial R \\
\partial R / \partial B_0 & \partial R / \partial B_1 & \partial R / \partial B_2 & \partial R / \partial R
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\left(3B_i - 2B_0\right) & \left(2B_i - 6B_0\right) & \left(6B_i - 3B_0\right) & \left(2B_i - 3B_0\right)
\end{bmatrix}
\] \hspace{1cm} \text{(B12)}

Then the variance-covariance matrix in the 1st transformation, i.e. of the 2nd column term in (B11) can be obtained by the right hand side of Eq. (27), where \( \sum_0 \) is the variance-covariance matrix \( B \).

\[
\sum_1 = J_1^T \sum_0 J_1 = \frac{1}{N} \begin{bmatrix}
B_{00} & B_{01} & B_{02} & C_0 \\
B_{10} & B_{11} & B_{12} & C_1 \\
B_{20} & B_{21} & B_{22} & C_2 \\
C_0 & C_1 & C_2 & C
\end{bmatrix}
\] \hspace{1cm} \text{(B13)}

where, the elements are defined in Eqs. (B14) through (B17).

\[
C_0 = \left\{ \left(3B_i - 2B_0\right)B_{00} + \left(2B_i - 6B_0\right)B_{01} + \left(6B_i - 3B_0\right)B_{02}\right\} / \left(2B_i - B_0\right)^2
\] \hspace{1cm} \text{(B14)}

\[
C_1 = \left\{ \left(3B_i - 2B_0\right)B_{10} + \left(2B_i - 6B_0\right)B_{11} + \left(6B_i - 3B_0\right)B_{12}\right\} / \left(2B_i - B_0\right)^2
\] \hspace{1cm} \text{(B15)}

\[
C_2 = \left\{ \left(3B_i - 2B_0\right)B_{20} + \left(2B_i - 6B_0\right)B_{21} + \left(6B_i - 3B_0\right)B_{22}\right\} / \left(2B_i - B_0\right)^2
\] \hspace{1cm} \text{(B16)}

\[
C = \left\{ \left(3B_i - 2B_0\right)^2 B_{00} + \left(2B_i - 6B_0\right)^2 B_{01} + \left(6B_i - 3B_0\right)^2 B_{02} + \left(3B_i - 2B_0\right)(2B_i - 6B_0)(6B_i - 3B_0) B_{02} + \left(3B_i - 2B_0\right)(2B_i - 6B_0)(6B_i - 3B_0) B_{12} + \left(3B_i - 2B_0\right)(6B_i - 3B_0)(2B_i - 6B_0) B_{22}\right\} / \left(2B_i - B_0\right)^4
\] \hspace{1cm} \text{(B17)}

2) 2ND TRANSFORMATION

For the 2nd transformation, the Jacobian matrix is

\[
\begin{bmatrix}
\partial \hat{B}_0 / \partial \hat{B}_0 & \partial \hat{B}_0 / \partial \hat{B}_1 & \partial \hat{B}_0 / \partial \hat{B}_2 & \partial \hat{B}_0 / \partial \hat{R} \\
\partial \hat{B}_1 / \partial \hat{B}_0 & \partial \hat{B}_1 / \partial \hat{B}_1 & \partial \hat{B}_1 / \partial \hat{B}_2 & \partial \hat{B}_1 / \partial \hat{R} \\
\partial \hat{B}_2 / \partial \hat{B}_0 & \partial \hat{B}_2 / \partial \hat{B}_1 & \partial \hat{B}_2 / \partial \hat{B}_2 & \partial \hat{B}_2 / \partial \hat{R} \\
\partial \hat{R} / \partial \hat{B}_0 & \partial \hat{R} / \partial \hat{B}_1 & \partial \hat{R} / \partial \hat{B}_2 & \partial \hat{R} / \partial \hat{R}
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \partial \hat{k} / \partial \hat{R}
\end{bmatrix}
\] \hspace{1cm} \text{(B18)}
where \( R = \frac{\Gamma(3-k)-2\Gamma(1-k)}{2\Gamma(2-k)-2\Gamma(1-k)} \)

Thus, the variance-covariance matrix is given by

\[
\sum_3 = J_3 \sum_2 J_3^T = \frac{1}{N} \begin{bmatrix}
B_{00} & B_{01} & B_{02} & EC_0 \\
B_{10} & B_{11} & B_{12} & EC_1 \\
B_{20} & B_{21} & B_{22} & EC_2 \\
EC_0 & EC_1 & EC_2 & E^2 C
\end{bmatrix}
\] (B19)

where \( E = \frac{\partial \hat{k}}{\partial R} = [2\Gamma(2-k)-2\Gamma(1-k)] I / \left\{ \begin{array}{c} 2\Gamma(1-k)\psi(1-k)-\Gamma(3-k)\psi(3-k) \\ \Gamma(3-k)-2\Gamma(1-k) \\ 2\Gamma(2-k)-2\Gamma(1-k)\psi(2-k) \end{array} \right\} \)

3) 3RD TRANSFORMATION

The Jacobian matrix in the 3rd transformation is

\[
J_3 = \begin{bmatrix}
\frac{\partial \hat{e}}{\partial \hat{B}_o} & \frac{\partial \hat{e}}{\partial \hat{B}_1} & \frac{\partial \hat{e}}{\partial \hat{B}_2} & \frac{\partial \hat{e}}{\partial \hat{e}} \\
\frac{\partial \hat{a}}{\partial \hat{B}_o} & \frac{\partial \hat{a}}{\partial \hat{B}_1} & \frac{\partial \hat{a}}{\partial \hat{B}_2} & \frac{\partial \hat{a}}{\partial \hat{e}} \\
\frac{\partial \hat{\alpha}}{\partial \hat{B}_o} & \frac{\partial \hat{\alpha}}{\partial \hat{B}_1} & \frac{\partial \hat{\alpha}}{\partial \hat{B}_2} & \frac{\partial \hat{\alpha}}{\partial \hat{e}} \\
\frac{\partial \hat{\psi}}{\partial \hat{B}_o} & \frac{\partial \hat{\psi}}{\partial \hat{B}_1} & \frac{\partial \hat{\psi}}{\partial \hat{B}_2} & \frac{\partial \hat{\psi}}{\partial \hat{e}}
\end{bmatrix} = \begin{bmatrix}
F_o & F_i & 0 & F_s \\
G_o & G_i & 0 & G_s \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (B20)

where the elements of matrix are given by

\[
F_o = \{1 - \Gamma(1+k)\Gamma(2-k)\} / \{\Gamma(1+k) - \Gamma(2-k)\} \] (B21)

\[
F_i = -2 \{1 - \Gamma(1+k)\Gamma(1-k)\} / \{\Gamma(1+k) - \Gamma(2-k)\} \] (B22)

\[
F_s = \alpha \Gamma(1+k)\Gamma(1-k)\psi(1-k) / k \] (B23)

\[
G_o = -k / \{\Gamma(1+k) - \Gamma(2-k)\} \] (B24)

\[
G_i = 2k / \{\Gamma(1+k) - \Gamma(2-k)\} \] (B25)

\[
G_s = \alpha \{1/k - \psi(1+k)\} - \{1/k - \psi(2-k)\} / \{\Gamma(1-k) - \Gamma(2-k)\} \] (B26)

Then the variance-covariance matrix can be obtained by

\[
\sum_3 = J_3 \sum_2 J_3^T = \frac{1}{N} \begin{bmatrix}
Z_{\alpha\alpha} & Z_{\alpha\alpha} & Z_{\alpha\alpha} \\
Z_{\alpha\alpha} & Z_{\alpha\alpha} & Z_{\alpha\alpha} \\
Z_{\alpha\alpha} & Z_{\alpha\alpha} & Z_{\alpha\alpha}
\end{bmatrix}
\] (B27)

where the diagonal and off-diagonal terms are the variances and covariances of estimators, respectively, given by Eqs. (39) through (44).

4) 4TH TRANSFORMATION

The quantile estimator of the GLO distribution for return period \( T \) is Eq. (8) and the Jacobian matrix is given by

\[
J_4 = \begin{bmatrix}
\frac{\partial \hat{x}_1}{\partial \hat{\psi}} & \frac{\partial \hat{x}_2}{\partial \hat{\psi}} & \frac{\partial \hat{x}_3}{\partial \hat{\psi}} \\
\frac{\partial \hat{x}_1}{\partial \hat{\psi}} & \frac{\partial \hat{x}_2}{\partial \hat{\psi}} & \frac{\partial \hat{x}_3}{\partial \hat{\psi}}
\end{bmatrix}
\] (B28)

where each element is given in Eqs. (36) through (38). Thus,

\[
\hat{S}_4 = J_4 \sum_3 J_4^T
\] (B29)

And the result is given in Eq. (45).